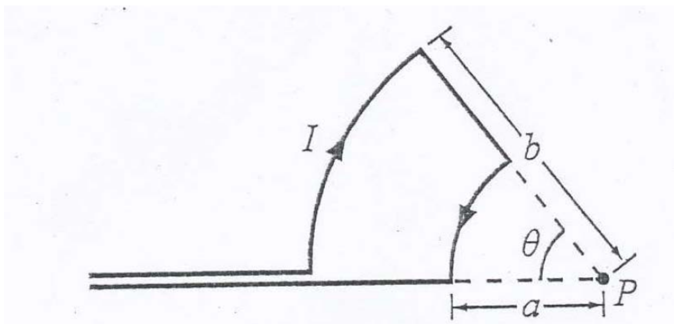
ECED 3300
Tutorial 8

Problem 1

The loop shown in the figure below consists of radial lines and segments of circles whose centers are located at point P.



Determine the magnetic field $\mathbf{H}$ at P in terms of a, b, θ, and I.

Solution

We use the Bio-Savart law; for a current element the law implies that

$$d\mathbf{H} = \frac{Id\mathbf{l} \times \mathbf{R}}{4\pi R^3}.$$

The loop consists of radial and semi-circular segments. As the problem is two-dimensional and has a clear-cut polar symmetry, we choose the polar coordinate system with the origin at P. Along the radial segments, $\phi = 0$ and $\phi = \theta$, $\mathbf{R} = \rho\mathbf{a}_\rho$ and $Id\mathbf{l} = Id\rho\mathbf{a}_\rho$ such that $Id\mathbf{l} \times \mathbf{R} = 0$. Thus there is no contribution to $\mathbf{H}$ at P due to radial segments. We now treat each circular segment separately.

(i) $r = a$: $Id\mathbf{l} = -Iad\phi\mathbf{a}_\phi$; the minus sign enters because the current direction is counterclockwise along this segment. Next, $\mathbf{R} = a\mathbf{a}_\rho$; putting together,

$$d\mathbf{H}_1 = -\frac{Ia^2 d\phi(\mathbf{a}_\phi \times \mathbf{a}_\rho)}{4\pi a^3},$$

It then follows that

$$\mathbf{H}_1 = -\frac{I}{4\pi a}\int_0^\theta d\phi\, \underbrace{(\mathbf{a}_\phi \times \mathbf{a}_\rho)}_{=-\mathbf{a}_z} = \frac{I\theta}{4\pi a}\mathbf{a}_z.$$

(ii) On $r = b$: $Id\mathbf{l} = Ibd\phi\mathbf{a}_\phi$ and $\mathbf{R} = b\mathbf{a}_\rho$, and by the same token as above,

$$\mathbf{H}_2 = -\frac{I\theta}{4\pi b}\mathbf{a}_z.$$

The overall field at P is given by the superposition principle by

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 = \frac{I\theta(b-a)}{4\pi ab}\mathbf{a}_z.$$

Problem 2

Given the magnetic field in a free space,

$$\mathbf{H} = xy^2\mathbf{a}_x + x^2z\mathbf{a}_y - y^2z\mathbf{a}_z, \textit{A/m},$$

determine

(i) the current density at P(2,-1,3);

(ii) the time rate of change of the electric charge at P.

Solution

(i) Using the Maxwell equation, $\mathbf{J} = \nabla \times \mathbf{H}$, we obtain

$$\mathbf{J} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial_x & \partial_y & \partial_z \\ xy^2 & x^2z & -y^2z \end{vmatrix}.$$

Working out the determinant, we obtain $\mathbf{J} = -(x^2 + 2yz)\mathbf{a}_x + 2x(z-y)\mathbf{a}_z$. At the point P, we have, $\mathbf{J} = -\mathbf{a}_x + 16\mathbf{a}_z$, A/m^2.

(ii) Using the continuity equation, $\partial_t\rho_v = -\nabla \cdot \mathbf{J} = -\partial_x J_x - \partial_y J_y - \partial_z J_z = 2x - 2x = 0$.

Problem 3

A very long, straight conducting cylinder of radius R is placed along the z-axis. There is a nonuniform current distribution $\mathbf{J}(\rho) = J_0(\rho/R)^2\mathbf{a}_z$, (A/m^2), where J_0 is a constant, across the interior of the cylinder. Determine the magnetic field $\mathbf{H}$ inside and outside of the cylinder. You may neglect any edge effects.

Solutions

Consider separately the interior and exterior of the cylinder.

The region $0 < \rho < R$. Applying Ampère's law,

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot d\mathbf{S},$$

which in this case simplifies to

$$H 2\pi\rho = \int \left(\frac{J_0 \rho^2}{R^2} \mathbf{a}_z \right) \underbrace{\rho d\rho d\phi \mathbf{a}_z}_{d\mathbf{S}} = \frac{J_0}{R^2} \int_0^{2\pi} d\phi \int_0^{\rho} \rho^3 d\rho = \frac{\pi J_0 \rho^4}{2R^2}. \tag{1}$$

Finally,

$$\mathbf{H} = \frac{J_0 \rho^3}{4R^2} \mathbf{a}_\phi. \tag{2}$$

The region $\rho > R$. The solution proceeds in exactly the same fashion, except the integration over ρ goes from 0 to R as the total current through the interior of the cylinder contributes in this case. The result is

$$\mathbf{H} = \frac{J_0 R^2}{4\rho} \mathbf{a}_\phi. \tag{3}$$